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Quantum mechanics on non commutative spaces and squeezed states: a functional approach

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Abstract: We review here the quantum mechanics of some noncommutative theories in which no state saturates simultaneously all the non trivial Heisenberg uncertainty relations. We show how the difference of structure between the Poisson brackets and the commutators in these theories generically leads to a harmonic oscillator whose positions and momenta mean values are not strictly equal to the ones predicted by classical mechanics.
This raises the question of the nature of quasi classical states in these models. We propose an extension based on a variational principle. The action considered is the sum of the absolute values of the expressions associated to the non trivial Heisenberg uncertainty relations. We first verify that our proposal works in the usual theory i.e. we recover the known gaussian functions. Besides them, we find other states which can be expressed as products of gaussians with specific hyper geometrics.
We illustrate our construction in two models defined on a four dimensional phase space: a model endowed with a minimal length uncertainty and the non commutative plane. Our proposal leads to second order partial differential equations. We find analytical solutions in specific cases. We briefly discuss how our proposal may be applied to the fuzzy sphere and analyze its shortcomings.

Keywords: Quantum Groups, Non-Commutative Geometry.

Contents

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1. Introduction

Many works have been devoted to noncommutative quantum theories recently. One of the main motivations is the hope that a non trivial structure of space time at small distances may give birth to theories with better ultraviolet behaviors.

There are two ways of apprehending non commutative theories. The first one postulates a modification of the commutation relations from the start. This is the case for example of J. Madore [1]. The second one, in the wake of E. Witten and N. Seiberg [2], considers non commutative theories as low energy phenomenological implications of string theory. The philosophy of this work relies on the first approach.

Many non commutative quantum theories have not fulfilled the initial hope concerning the convergence of Green functions [3]. One of the remarkable cases which do not fall in this category has been formulated by Kempf-Mangano-Mann(KMM) [4]–[6]. This could be achieved thanks to a careful analysis of the states physically allowed in the theory. With that idea in mind, it was suggested that the analysis about the loss of causality may need a more detailed treatment [7].

Some efforts have been devoted to the understanding of quantum mechanics when non commutativity sets in. For example, H.Falomir et al. [8] have studied the Bohm-Aharonov effect in this context, obtaining for the deformation parameter new bounds which are consistent with previous ones [9]. The noncommutative oscillator in arbitrary dimension has been analyzed by A.Hatzinikitas et al. [10] while R. Banarjee [11] has shown its link to dissipation. The relation with usual canonical variables has been analyzed by A. Smailagic and E. Spallucci [12] and a path integral formulation proposed by C. Acatrinei [13]. Some theories which also have special fundamental structures have been used in the study of physical processes like black hole evaporation [14, 15].

Our aim is not to solve a specific problem in this framework but to try to understand the structure of some meaningful sets of states. In a recent work, K. Bolonek and P. Kosinski [16] have shown that all the non trivial Heisenberg uncertainties can not be satisfied simultaneously in a theory where the commutators of the positions are non vanishing constants. This raises an important question since in the unmodified theory, the states which saturate all the non trivial Heisenberg uncertainties are also the ones which lead to classical trajectories. In these states, the mean values of the position and momentum operators reproduce the behavior of the classical Hamilton solution for the harmonic oscillator, the relative uncertainties being negligible [17].

Adapting for our purposes an idea recently used by S. Detournay, Cl. Gabriel and Ph. Spindel [18], we construct a functional which attains its minima on the coherent states in the usual theory. We then generalize this functional for the non commutative theories we are interested in. The minima of these functionals are, in our opinion, candidates to the status of squeezed states. Our work is somehow similar in spirit with the paper by G. Dourado Barbosa [19] but the approach and the tools are different.

This paper is organized as follows. The section 2 is devoted to the impossibility of saturating the non trivial Heisenberg uncertainties in two particular noncommutative theories. In the section 3 we show that for a generic state of a harmonic oscillator on the non commutative plane or in the KMM model, the mean positions do not reproduce rigorously the behavior obtained using a classical analysis; although experimentally unobservable, this fact raises an important conceptual question. In the section 4, we show explicitly how our proposal works in the usual one dimensional case; we verify that the usual squeezed states satisfy the second order differential equation which comes from our variational principle. In the section 5 we apply the same method to the two models studied in section 2. We obtain that the states for which the functionals are extremal verify second order partial differential equations. We give some exact solutions but it remains to be proved that they effectively realize the minimum of the functional. In the section 6 we briefly discuss how our procedure can be extended to the fuzzy sphere. The section 7 is devoted to a discussion of our results.

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2. Examples of high dimensional theories with non trivial Heisenberg uncertainties that cannot be satisfied simultaneously

In usual quantum theory, the coherent states satisfy *simultaneously* the relations

$$\Delta x_i \Delta p_i = \frac{|\langle[\hat{x}_i, \hat{p}_i]\rangle|}{2} = \frac{\hbar}{2} \tag{2.1}$$

for each index i. On the contrary, a product such as $\Delta x_1 \Delta p_2$ can be arbitrary since the commutator of the associated variables vanishes. The first type of uncertainty will be called *non trivial*. The aim of this section is to show that in some two dimensional models all the non trivial uncertainties cannot be saturated at the same time. The same reasoning works in higher dimensions.

To proceed, we will associate to each fixed state $|\psi\rangle$ an operator $\hat{a}_\psi$ whose action on an arbitrary state $|\phi\rangle$ is defined by

$$\hat{a}_\psi|\phi\rangle = \left[\hat{x} - \langle\hat{x}\rangle_\psi I + \frac{\langle[\hat{x},\hat{p}]\rangle_\psi}{2(\Delta_\psi p)^2}(\hat{p} - \langle\hat{p}\rangle_\psi I)\right]|\phi\rangle\,, \tag{2.2}$$

I being the identity operator. The states $|\psi\rangle$ which satisfy eq. (2.1) and thus saturate the Heisenberg uncertainty obey

$$\hat{a}_\psi|\psi\rangle = 0\,. \tag{2.3}$$

In practice one fixes the state $|\psi\rangle$, solves the *differential equation* obtained by equating the expression of eq. (2.2) to zero and then retains the solution if it satisfies $|\psi\rangle = |\phi\rangle$. We use here a harmless abuse of language: assigning numerical values to $\langle\hat{x}\rangle_\psi$ and $\frac{\langle[\hat{x},\hat{p}]\rangle_\psi}{2(\Delta_\psi p)^2}\langle\hat{p}\rangle_\psi$ does not completely fix the state $|\psi\rangle$.

2.1 The non commutative plane

Let us consider a two dimensional theory in which the non vanishing commutation relations are the following [16]:

$$[\hat{x}_j, \hat{x}_k] = i\epsilon_{jk}\,\theta\,, \qquad [\hat{x}_j, \hat{p}_k] = i\,\hbar\,\delta_{jk}\,; \qquad \theta, \hbar > 0\,. \tag{2.4}$$

This theory admits a representation in which the operators and the scalar product are given by the following formula

$$\hat{x}_1 = i\hbar\partial_{p_1} - \frac{1}{2}\frac{\theta}{\hbar}p_2\,, \quad \hat{x}_2 = i\hbar\partial_{p_2} + \frac{1}{2}\frac{\theta}{\hbar}p_1\,, \quad \hat{p}_1 = p_1\,, \quad \hat{p}_2 = p_2\,, \quad \langle\phi|\psi\rangle = \int d^2p\,\phi^*(p)\psi(p)\,. \tag{2.5}$$

As reminded above, the equality in the Heisenberg relation is attained only for those states which satisfy eq. (2.3). The operator defined in eq. (2.2) is built from $\hat{x}$ and $\hat{p}$. Similarly, having two arbitrary operators $\hat{u}, \hat{v}$ and a state $|\psi\rangle$ one can define a third operator

$$\hat{a} = \hat{u} + i\lambda\hat{v} + \mu I \quad \text{where} \quad \lambda = \frac{\langle[\hat{u},\hat{v}]\rangle_\psi}{2(\Delta_\psi v)^2} \quad \text{and} \quad \mu = -\langle\hat{u}\rangle_\psi - i\frac{\langle[\hat{u},\hat{v}]\rangle_\psi}{2(\Delta_\psi v)^2}\langle\hat{v}\rangle_\psi\,. \tag{2.6}$$

In our case,

$$\begin{aligned}
\Delta x_1 \Delta x_2 &= \frac{\theta}{2} \Longrightarrow \hat{a}_{1\psi}|\psi\rangle = 0\,, \\
\Delta x_1 \Delta p_1 &= \frac{\hbar}{2} \Longrightarrow \hat{a}_{2\psi}|\psi\rangle = 0\,, \\
\Delta x_2 \Delta p_2 &= \frac{\hbar}{2} \Longrightarrow \hat{a}_{3\psi}|\psi\rangle = 0\,;
\end{aligned} \tag{2.7}$$

the operators $\hat{a}_i$ are given by the following expressions

$$\hat{a}_{1\psi} = \hat{x}_1 + i\lambda_1 \hat{x}_2 + \mu_1 I\,, \qquad \hat{a}_{2\psi} = \hat{x}_1 + i\lambda_2 \hat{p}_1 + \mu_2 I\,, \qquad \hat{a}_{3\psi} = \hat{x}_2 + i\lambda_3 \hat{p}_2 + \mu_3 I\,. \tag{2.8}$$

Looking for states $|\phi\rangle$ verifying $\hat{a}_{1\psi}|\phi\rangle = \cdots = 0$, the quantities λ_i and μ_i become constants. The second and the third equalities in eq. (2.7) can not be satisfied simultaneously. This is due to the fact that if a state satisfying this exists, it should be in the kernel of the two corresponding operators. Then one would have

$$[\hat{a_{2}}_{\psi}, \hat{a_{3}}_{\psi}]|\phi\rangle = i\theta|\phi\rangle = 0 \tag{2.9}$$

which admits only the null vector as a solution. A more extensive analysis of the other equalities can be found in [16].

The "constants" μ_k appearing in the operators $\hat{a}_i$ in eq. (2.8) do not play any role in the considerations implying the commutators. We shall discard them for simplicity in the model considered below. For simplicity, the real quantities λ_k will be treated as "constants" in what follows; that means the state $|\psi\rangle$ has been fixed and one is solving the equation for $|\phi\rangle$. Due to this convention, we shall not write a subscript to our operators explicitly.

2.2 The two dimensional model with a minimal uncertainty in length

The first extension of the one dimensional model endowed with a minimal length uncertainty was proposed in [20]:

$$[\hat{x}_j, \hat{p}_k] = i\hbar(1 + \beta\hat{\vec{p}}^2)\delta_{jk}\,, \qquad [\hat{x}_j, \hat{x}_k] = 2i\hbar\beta(\hat{p}_j \hat{x}_k - \hat{p}_k \hat{x}_j)\,. \tag{2.10}$$

It lacks translational symmetry since the second relation is not invariant under the transformation $\hat{x}_k \to \hat{x}_k + \hat{\alpha}_k I$, $\hat{p}_k \to \hat{p}_k$. The second extension of the KMM model to higher dimensions is invariant under rotations and translations. The non trivial commutation relations are [4]:

$$[\hat{x}_j, \hat{p}_k] = i\hbar\left(f(\hat{\vec{p}}^2)\delta_{jk} + g(\hat{\vec{p}}^2)\hat{p}_j\hat{p}_k\right), \tag{2.11}$$

and the condition

$$g = \frac{2ff'}{f - 2p^2 f'}\,, \tag{2.12}$$

enforces the commutation of the positions among themselves. The same property holds for the momenta. The functions f and g are supposed positive. In this model, a position operator in one direction has a non trivial commutation relation with the momentum associated to another direction:

$$[\hat{x}_1, \hat{p}_2] = ihg(\hat{\vec{p}}^2)\hat{p}_1\hat{p}_2\,. \tag{2.13}$$

A representation of this extension is realized by the following formulas:

$$\hat{x}_i = i\hbar \left(f' + \vec{p}^2 g' + \frac{3}{2} g \right) p_i + f\partial_{p_i} + g p_i p_j \partial_{p_j} \,, \qquad \hat{p}_i = p_i \,, \quad \langle \phi | \psi \rangle = \int d^2 p \phi^*(p) \psi(p) \,. \tag{2.14}$$

In a space of dimension two, this theory has four non trivial uncertainty relations concerning the following couples of variables: $(x_1, p_1), (x_2, p_2), (x_1, p_2), (x_2, p_1)$. The corresponding operators are

$$\hat{a}_1 = \hat{x}_1 + i\lambda \hat{p}_1 \,, \qquad \hat{a}_2 = \hat{x}_2 + i\mu \hat{p}_2 \,, \qquad \hat{a}_3 = \hat{x}_1 + i\tau \hat{p}_2 \,, \qquad \hat{a}_4 = \hat{x}_2 + i\sigma \hat{p}_1 \,, \tag{2.15}$$

so that the relevant commutators, are, in this case

$$\begin{aligned}
[\hat{a}_1, \hat{a}_2] &= \hbar g(\hat{\vec{p}}^2)(-\mu \hat{p}_1 \hat{p}_2 + \lambda \hat{p}_2 \hat{p}_1) \,, \\
[\hat{a}_1, \hat{a}_3] &= -\hbar \left(g(\hat{\vec{p}}^2)(\tau \hat{p}_1 \hat{p}_2 - \lambda \hat{p}_1^2) - \lambda f(\hat{\vec{p}}^2) \right) \,, \\
[\hat{a}_1, \hat{a}_4] &= -\hbar \left(g(\hat{\vec{p}}^2)(-\lambda \hat{p}_2 \hat{p}_1 + \sigma \hat{p}_1^2) + \sigma f(\hat{\vec{p}}^2) \right) \,, \\
[\hat{a}_2, \hat{a}_3] &= -\hbar \left(g(\hat{\vec{p}}^2)(-\mu \hat{p}_1 \hat{p}_2 + \tau \hat{p}_2^2) + \tau f(\hat{\vec{p}}^2) \right) \,, \\
[\hat{a}_2, \hat{a}_4] &= -\hbar \left(g(\hat{\vec{p}}^2)(\sigma \hat{p}_2 \hat{p}_1 - \mu \hat{p}_2^2) - \mu f(\hat{\vec{p}}^2) \right) \,, \\
[\hat{a}_3, \hat{a}_4] &= -\hbar \left(g(\hat{\vec{p}}^2)(\sigma \hat{p}_1^2 - \tau \hat{p}_2^2) + (\sigma - \tau) f(\hat{\vec{p}}^2) \right) \,.
\end{aligned} \tag{2.16}$$

From now on, we use the form of the operators given earlier. The first commutator has a non trivial kernel for $\lambda = \mu$. In particular, there are non zero vectors $|\psi\rangle$ fulfilling $\lambda = \mu$ which also fulfill $\hat{a}_{2_\psi}|\psi\rangle = \hat{a}_{3\psi}|\psi\rangle = 0$, and saturate simultaneously the uncertainty relations concerning the couples of variables (x_1, p_1) and (x_2, p_2). These states are the ones used by A. Kempf [20], S.Detournay Cl. Gabriel and Ph. Spindel [18] while studying the existence of a minimal uncertainty in length. We are interested in a conceptually different problem: *in this work we put all the non trivial commutation relations on the same footing.*

The action of the second commutator on a state $|\phi\rangle$ vanishes only if the equation

$$\left((\tau p_1 p_2 - \lambda p_1^2) - \lambda \frac{f(\vec{p}^2)}{f(\vec{p}^2)} \right) \phi(\vec{p}) = 0 \tag{2.17}$$

is satisfied. Clearly this is the case only when the expression under parentheses vanishes. As its second term is a function of $\vec{p}^2$, one should have the same thing for its first term. This is possible only if $\tau = \lambda = 0$, but since at the end of the analysis

$$\lambda = \frac{\hbar}{2(\Delta p_1)^2} \int d^2 p \phi^*(\vec{p}) \left(f(\vec{p}^2) + g(\vec{p}^2) p_1^2 \right) \phi(p) \,, \tag{2.18}$$

and the functions f, g are positive, this is excluded.

3. Periodicity of the harmonic oscillator

The coherent states, in the usual case, are not only the states which saturate simultaneously all the non trivial uncertainties; they also reproduce the classical behavior of the harmonic oscillator. In this section we will interest ourselves to the mean values of the positions and momenta of a harmonic oscillator in the theories under study. In usual quantum theory, any state which is a solution of the Schrodinger equation for the harmonic oscillator displays periodic positions and momenta; moreover the period coincides exactly with the one found in classical mechanics [17]. We will see that this is generically not the case when non commutativity sets in.

3.1 The non commutative plane

Working in the Heisenberg picture, the states are time independent while the operators vary according to the evolution equation

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{p}}_1 \\ \dot{\hat{p}}_2 \end{pmatrix} = \begin{pmatrix} 0 & k\theta/\hbar & \frac{1}{m} & 0 \\ -k\theta/\hbar & 0 & 0 & 1/m \\ -k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{p}_1 \\ \hat{p}_2 \end{pmatrix} . \tag{3.1}$$

This system of differential equations is easily solved using the exponential method. For example, one has for the first position

$$\hat{x}_1(t) = M_{11}(t)\hat{x}_1(0) + M_{12}(t)\hat{x}_2(0) + M_{13}(t)\hat{p}_1(0) + M_{14}(t)\hat{p}_2(0) , \tag{3.2}$$

with

$$\begin{aligned} M_{11}(t) &= \frac{1}{\lambda_1+\lambda_2}(\lambda_1 \cos\lambda_1 t + \lambda_2 \cos\lambda_2 t) , & M_{12}(t) &= \frac{1}{\lambda_1+\lambda_2}(-\lambda_1 \sin\lambda_1 t + \lambda_2 \sin\lambda_2 t) \\ M_{13}(t) &= \frac{\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}\frac{\theta}{\hbar}(-\sin\lambda_1 t - \sin\lambda_2 t) , & M_{14}(t) &= \frac{\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}\frac{\theta}{\hbar}(-\cos\lambda_1 t + \cos\lambda_2 t) . \end{aligned} \tag{3.3}$$

The eigenvalues of the matrix appearing in eq. (3.1) are $\pm i\lambda_1$and $\pm i\lambda_2$ where

$$\lambda_1 = \sqrt{\frac{k}{m} + \frac{k^2\theta^2}{2\hbar^2} - \frac{k^{3/2}\theta}{2\hbar^2\sqrt{m}}\sqrt{4\hbar^2 + km\theta^2}} , \quad \lambda_2 = \sqrt{\frac{k}{m} + \frac{k^2\theta^2}{2\hbar^2} + \frac{k^{3/2}\theta}{2\hbar^2\sqrt{m}}\sqrt{4\hbar^2 + km\theta^2}} \tag{3.4}$$

are real numbers. For a given wave function, the operators evaluated at the initial time (like $\langle\hat{x}_1(0)\rangle$) are obtained thanks to the expressions displayed in eq. (2.5). The mean value of the first position in any state $\psi(p_1, p_2)$ can be rewritten as

$$\langle\hat{x}_1(t)\rangle = \frac{1}{\hbar(\lambda_1-\lambda_2)(\lambda_1+\lambda_2)}(c_1 \cos\lambda_1 t + s_1 \sin\lambda_1 t + c_2 \cos\lambda_2 t + s_2 \sin\lambda_2 t) , \tag{3.5}$$

with

$$\begin{aligned} c_1 &= \hbar\langle\hat{x}_1(0)\rangle\lambda_1^2 - \lambda_1\lambda_2(\hbar\langle\hat{x}_1(0)\rangle - \theta\langle\hat{p}_2(0)\rangle) , \\ s_1 &= -\hbar\langle\hat{x}_2(0)\rangle\lambda_1^2 + \lambda_1\lambda_2(\hbar\langle\hat{x}_2(0)\rangle - \theta\langle\hat{p}_1(0)\rangle) , \\ c_2 &= \hbar\langle\hat{x}_1(0)\rangle\lambda_2^2 - \lambda_1\lambda_2(\hbar\langle\hat{x}_1(0)\rangle + \theta\langle\hat{p}_2(0)\rangle) , \\ s_2 &= \hbar\langle\hat{x}_2(0)\rangle\lambda_2^2 + \lambda_1\lambda_2(-\hbar\langle\hat{x}_2(0)\rangle + \theta\langle\hat{p}_1(0)\rangle) . \end{aligned} \tag{3.6}$$

From eq. (3.5), one sees that the position mean value will be periodic for all states only if the ratio of the two frequencies λ_1/λ_2 is rational. This requires a fine tuning of the parameters and so is not generic. In the cases where the ratio is not rational, there are states for which the coefficients are such that the components of one of the frequencies vanish. For example, the component of frequency λ_1 disappears in $\langle\hat{x}_1(t)\rangle$ if the following conditions are satisfied:

$$\theta\lambda_2\langle\hat{p}_1(0)\rangle + \hbar(\lambda_1-\lambda_2)\lambda_2\langle\hat{x}_2(0)\rangle = 0\,, \quad \theta\lambda_2\langle\hat{p}_2(0)\rangle + \hbar(-\lambda_1+\lambda_2)\lambda_2\langle\hat{x}_1(0)\rangle = 0\,. \tag{3.7}$$

Moreover, these two conditions also suppress the λ_1 components in the mean values of the remaining variables x_2, p_1, p_2. The above conditions translate into the vanishing of the following integrals:

$$\begin{aligned} &\int d^2p\psi^*(p)\left(i\hbar^2(\lambda_1-\lambda_2)\partial_{p_2} + \frac{1}{2}\theta(\lambda_1+\lambda_2)p_1\right)\psi(p)\,, \\ &\int d^2p\psi^*(p)\left(i\hbar^2(\lambda_1-\lambda_2)\partial_{p_1} - \frac{1}{2}\theta(\lambda_1+\lambda_2)p_2\right)\psi(p)\,. \end{aligned} \tag{3.8}$$

It is readily found that the operators contained in the parentheses do not have a common zero eigenvalue so that the states we are looking for cannot be found by solving two first order differential equations. However, one can show that the wave function

$$\psi(p) = \exp\left((-a_1^2+ia_2)p_1^2 + (-b_1^2+ib_2)p_2^2 + (c_1+ic_2)p_1 + (d_1+id_2)p_2\right) \tag{3.9}$$

satisfies the two conditions provided that the following relations hold:

$$c_2 = -\frac{a_2c_1}{a_1^2} - \frac{1}{4}\frac{\theta}{\hbar^2}\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\frac{d_1}{b_1^2}\,, \qquad d_2 = -\frac{b_2d_1}{b_1^2} + \frac{1}{4}\frac{\theta}{\hbar^2}\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\frac{c_1}{a_1^2}\,. \tag{3.10}$$

It can be easily seen from eqs. (3.4), (3.5) that the usual theory is recovered when the deformation parameter goes to zero.

3.2 The model possessing a minimal length uncertainty

For the KMM theory, the situation is more complicated and we will treat only the one dimensional case. The analysis in the Heisenberg picture is not straightforward because the evolution of the operators leads to non linear equations like

$$\dot{\hat{x}} = \frac{1}{m}(1+\beta\hat{p}^2)\hat{p}\,. \tag{3.11}$$

We will rather use the Schrodinger picture and the knowledge of the spectrum of the harmonic oscillator [20]. The operators are now fixed and the time dependence is carried by the state. The energy eigenstates form a basis of the Fock space and so an arbitrary state can be expanded as $|\psi\rangle = \sigma^n|n\rangle$. The mean value of the position in this state reads

$$\langle\hat{x}(t)\rangle = \sum_{m,n}\sigma^n(\sigma^*)^m\langle m|\hat{x}|n\rangle\exp\left(\frac{i}{\hbar}(E_m-E_n)t\right); \tag{3.12}$$

the time dependence is entirely contained in the exponentials. In the usual case the matrix element $\langle m|\hat{x}|n\rangle$ is non vanishing only for integers m, n which differ by one unit. As the energy spectrum is then linear, the arguments of the exponentials are all equal if we discard the signs. Collecting them, one ends up with the usual superposition of sine and cosine functions. In the KMM theory, the spectrum of the harmonic oscillator is quadratic [20]: $E_n = an^2 + bn + c$, where the constants a, b, c depend on the characteristics of the oscillator and the deformation parameter. This leads us to the following expression for the position mean value:

$$\langle \hat{x}(t) \rangle = \sum_{m,n} \tau_{m,n} (\cos \omega_{mn} t + i \sin \omega_{mn} t) , \qquad \omega_{mn} = \frac{1}{\hbar}(m-n)(a(m+n)+b) . \tag{3.13}$$

This mean value will be periodic only if the frequency ratio $\omega_{m_1 n_1}/\omega_{m_2 n_2}$ is a rational number every time $\tau_{m_1,n_1}, \tau_{m_2,n_2}$ are non vanishing. Then, there must exist a rational number r_{12} such that

$$r_{12} = \frac{a(m_1+n_1)+b}{a(m_2+n_2)+b} . \tag{3.14}$$

From this one infers that b/a must then be rational; replacing the quantities a and b in terms of the quantum parameters [20], this amounts to ask that

$$\frac{\beta m \hbar \omega}{\beta m \hbar \omega + \sqrt{4 + \beta m \hbar \omega}} \tag{3.15}$$

is rational. When this condition is not satisfied, periodicity is lost.

4. The least square variational principle in the usual 1D case

We have seen in the last sections that two defining properties of the coherent states (reproduction of the harmonic oscillator's classical behavior, saturation of the non trivial Heisenberg uncertainties) are lost as one goes to non commutative theories. The question we now face is: what possible generalization can one adopt? The first possibility relies on a deformation of the algebra generated by the creation-destruction operators: we will say a few words about it in the next section. The generalization we propose here strictly relies on the uncertainties. Let us consider a theory whose position and momentum operators are generically denoted $\hat{G}_i$ and whose non trivial uncertainty relations can not be satisfied simultaneously.

We will call quasi classical or squeezed states those which display a minimum of a functional of the type

$$S_{n,m} = \sum_{j,k} \left| (\Delta G_j)^{2n} (\Delta G_k)^{2n} - \frac{|\langle [\hat{G}_j, \hat{G}_k] \rangle|^{2n}}{2^{2n}} \right|^{2m} . \tag{4.1}$$

The sum runs on all couples (j,k) such that the commutators $[\hat{G}_j, \hat{G}_k]$ do not vanish. The values of the indices (n,m) in which we will be interested are 1/2 and 1; we shall come back to this point later. It is important that all choices of (n,m) lead essentially to the same differential equation when the action is varied, as we shall see. We restrict ourselves to pairs of variables with non vanishing commutators in order to recover the desired results in the unmodified theory for higher dimensions; this will be discussed at the end of this section.

In the usual theory, the action simply vanishes on squeezed states. The difficult point in our proposal lies in the fact that the variation of the sum given in eq. (4.1) leads to a second order differential equation which in principle admits more solutions. Let us show how this works in the simplest i.e undeformed theory. The action we consider is

$$S = \left| (\Delta x)^2 (\Delta p)^2 - \frac{1}{4} \right| , \tag{4.2}$$

where for simplicity we have set $\hbar = 1$. *The Heisenberg inequality tells us that we can remove the absolute value.* The definition of the uncertainty

$$(\Delta x)^2 = \frac{\langle\psi|\hat{x}^2|\psi\rangle}{\langle\psi|\psi\rangle} - \left(\frac{\langle\psi|\hat{x}|\psi\rangle}{\langle\psi|\psi\rangle}\right)^2 \tag{4.3}$$

renders necessary the computation of derivatives such as

$$\frac{\delta}{\delta\psi^*(p)}\langle\psi|\hat{x}|\psi\rangle = \frac{\delta}{\delta\psi^*(p)}\int dq\,\psi^*(q) i\partial_q \psi(q) = \int dq\,\delta(p-q) i\partial_q\psi(q) = i\partial_p\psi(p)\,. \tag{4.4}$$

Performing similar computations, we obtain the following equation when varying the action S:

$$\mathcal{O}|\psi\rangle = \langle\psi|\psi\rangle \frac{\delta S}{\delta\psi^*(p)} = a\psi^{''}(p) + ib\psi^{'}(p) + (cp^2 + dp + e)\psi(p) = 0\,, \tag{4.5}$$

where a, b, c, d, e are linked to the observables of the solution by the relations

$$\begin{aligned} a &= -(\Delta p)^2\,, \qquad b = -2(\Delta p)^2\langle\hat{x}\rangle\,, \qquad c = (\Delta x)^2\,, \qquad d = -2(\Delta x)^2\langle\hat{p}\rangle\,, \\ e &= -2(\Delta x)^2(\Delta p)^2 + (\Delta x)^2\langle\hat{p}\rangle^2 + (\Delta p)^2\langle\hat{x}\rangle^2\,. \end{aligned} \tag{4.6}$$

The mean values are taken on the same state $|\psi\rangle$. The appearance of the observables of the solution such as $\langle\hat{x}\rangle, \cdots$ in this equation is similar to the situation encountered in eq. (2.3) i.e for usual coherent states. One can similarly to eq. (2.2) define an operator $\mathcal{O}_\psi$ depending on a state $|\psi\rangle$ such that its action on any state $|\phi\rangle$ is given by

$$\mathcal{O}_\psi|\phi\rangle = -(\Delta_\psi p)^2\phi^{''}(p) - 2\,i\,(\Delta_\psi p)^2\,\langle\hat{x}\rangle_\psi\phi^{'}(p) + \tag{4.7}$$
$$+ [(\Delta_\psi x)^2 p^2 - 2(\Delta_\psi x)^2\langle\hat{p}\rangle_\psi p +$$
$$+ (-2(\Delta_\psi x)^2(\Delta_\psi p)^2 + (\Delta_\psi x)^2\langle\hat{p}\rangle_\psi^2 + (\Delta_\psi p)^2\langle\hat{x}\rangle_\psi^2)]\phi(p)\,, \tag{4.8}$$

leading to a differential equation.

Let us verify that the well known coherent states satisfy our equation; we look for gaussians [17]:

$$\phi(p) = N\exp\left((z_1 + iz_2)p^2 + (z_3 + iz_4)p\right) \tag{4.9}$$

which are solutions of

$$\mathcal{O}_\psi|\phi\rangle = 0\,. \tag{4.10}$$

This happens if the z_i are given by

$$z_1 = -\sqrt{-\frac{c}{4a}} \equiv -\sqrt{-\frac{(\Delta_\psi x)^2}{4(\Delta_\psi p)^2}}\,, \quad z_2 = 0\,, \quad z_3 = \frac{d}{2a}\sqrt{-\frac{a}{c}}\,, \quad z_4 = -\frac{b}{2a} \tag{4.11}$$

and the following relation between the parameters $a, \ldots, e$ holds:

$$e = -\frac{b^2}{4a} - \sqrt{-ac} + \frac{d^2}{4c}\,. \tag{4.12}$$

The final expression of z_1 in eq. (4.11) is obtained using eq. (4.6); the same thing can be done for the other z_k. Due to eq. (4.6), the z_i are real. They depend on the state $|\psi\rangle$. The normalization of the wave function to one is achieved by an appropriate choice of the pre factor N. One finds

$$\Delta_\phi x = \frac{1}{\sqrt{2}}\left(-\frac{c}{a}\right)^{1/4}, \qquad \Delta_\phi p = \frac{1}{\sqrt{2}}\left(-\frac{a}{c}\right)^{-1/4}. \tag{4.13}$$

Imposing that the states $|\phi\rangle$ and $|\psi\rangle$ coincide amounts to replace $\Delta_\phi x$ by $\Delta_\psi x$ in the left sides of eq. (4.13) and the parameters $a, \ldots, e$ by their expressions given by eq. (4.6). This leads to the equality

$$\Delta_\psi x \Delta_\psi p = \frac{1}{2}\,. \tag{4.14}$$

This relation could also be obtained by multiplying the two equalities given in eq. (4.13) or by developing eq. (4.12) using eq. (4.6). The action S vanishes on the state we have obtained. We have thus explicitly verified that the usual coherent states are captured by our procedure.

Instead of simply looking for the conditions under which the known gaussians obey eq. (4.5), one may look for the cases in which the operator of eq. (4.7) factorizes, leading to two first order operators: $\mathcal{O}_\psi = \hat{k}_1\hat{k}_2$. It is not difficult to show that this occurs when eq. (4.12) is satisfied. One then has

$$\begin{aligned} \hat{k}_1\phi(p) &= -i\sqrt{-a}\phi'(p) + \left(\frac{1}{2}\left(-\frac{b}{\sqrt{-a}} + \frac{id}{\sqrt{c}}\right) + i\sqrt{c}p\right)\phi(p) \\ \hat{k}_2\phi(p) &= -i\sqrt{-a}\phi'(p) + \left(\frac{1}{2}\left(-\frac{b}{\sqrt{-a}} - \frac{id}{\sqrt{c}}\right) - i\sqrt{c}p\right)\phi(p)\,. \end{aligned} \tag{4.15}$$

A state which is in the kernel of $\hat{k}_2$ is obviously in the kernel of $\mathcal{O}$. Using the expressions of the operators $\hat{x}, \hat{p}$, one finds

$$\hat{k_2} = (\Delta_\psi p)\left[\hat{x} - \langle\hat{x}\rangle_\psi I + i\left(\frac{\Delta_\psi x}{\Delta_\psi p}\right)(\hat{p} - \langle\hat{p}\rangle_\psi I)\right]. \tag{4.16}$$

The relation given in eq. (4.12) which allows factorization also ensures eq. (4.14) as stated before. This can then be used to show that the operator $\hat{k}_2$ is proportional to the first order operator appearing in eq. (2.2). We have thus seen how the usual first order differential equation which leads to squeezed states in the undeformed case can be recovered with the method we propose.

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So far we have studied very specific solutions to the second order equation we obtained. To find the most general solution to eq. (4.5), let us make the following change of variables:

$$\begin{aligned}\psi(p) &= \exp\left(\frac{-4ibp+(-a)^{1/2}c^{-3/2}(d+2cp)^2}{8a}\right)(d+2cp)X(q)\,;\\ q &= -\frac{1}{4}(-a)^{-1/2}c^{-3/2}(d+2cp)^2\,.\end{aligned}\tag{4.17}$$

To simplify the expressions, let us also introduce the quantity

$$\alpha = \frac{-(-a)^{1/2}b^2c+a(12ac^{3/2}+(-a)^{1/2}(d^2-4ce))}{16a^2c^{3/2}}\,.\tag{4.18}$$

The differential equation now becomes

$$X''(q)+\left(1+\frac{3}{2q}\right)X'(q)+\frac{\alpha}{q}X(q)=0\tag{4.19}$$

and its general solution can be written as a sum of hyper geometric functions:

$$X(q)=C_1\,q^{-1/2}{}_1F_1\left(-\frac{1}{2}+\alpha,\frac{1}{2},-q\right)+C_2{}_1F_1\left(\alpha,\frac{3}{2},-q\right)\,.\tag{4.20}$$

Using for the constant e the expression displayed in eq. (4.12), the first argument of the two hyper geometrics equals $1/2$. As the second hyper geometric becomes a constant in this case, one recovers the gaussian solution given in eq. (4.9) by taking $C_2=0$.

Apart from the states which are known to have a vanishing value of S, the formalism we have used introduces new states. The fact that the gaussians displayed in eq. (4.9) exhibit the absolute minimum of the action and form an over complete set will be enough to discard the other states.

It should be stressed that normalizable solutions do not exist for all the values of the parameters. In fact, the change of variables

$$\psi(p)=\exp\left(-\frac{ib}{2a}p\right)u(p)\tag{4.21}$$

gives to eq. (4.5) the following form

$$u''(p)+V(p)u(p)=0\,,\qquad V(p)=\left(\frac{c}{a}p^2+\frac{d}{a}p+\frac{e}{a}+\frac{b^2}{4a^2}\right)\,.\tag{4.22}$$

Integrating by parts and discarding the boundary term, one obtains the relation

$$\int_{-\infty}^{+\infty}dp\,\left(u'(p)^2-V(p)u(p)^2\right)=0\,,\tag{4.23}$$

which can not be satisfied unless the opposite of $V(p)$ admits two distinct zeros. This results in the following inequality:

$$e>\frac{d^2}{4c}-\frac{b^2}{4a}\,.\tag{4.24}$$

This reasoning applies when the function $u(p)$ is real. If it is complex, its real and imaginary parts obey eq. (4.22) whose parameters are real; the same conclusion holds.

To end this section, let us first notice that the choice $n = m = 1/2$ in eq. (4.2) can be chosen without changing drastically the situation. In fact, trivial relations such as

$$\frac{\delta}{\delta\psi^*(p)}\Delta x = \frac{1}{2\Delta x}\frac{\delta}{\delta\psi^*(p)}(\Delta x)^2 \tag{4.25}$$

show that if we work with $n = 1/2$, we will end up with a differential equation of the same form than eq. (4.5). The "only" difference will be encoded in a modified eq. (4.6) which gives the link between the parameters $a, \ldots, e$ and the observables of the state. What happens when one considers a different value of m? Then, the quantities $a, \ldots, e$ of eq. (4.6) are multiplied by the same factor $2m((\Delta_\psi x)^{2n}(\Delta_\psi p)^{2n} - 1/4^n)^{2m-1}$ *which vanishes* for the usual coherent states. The differential equation (4.7) then reduces, for this choice of $|\psi\rangle$, to the identity $0 = 0\,\phi(p)$. Nevertheless, the extra factor being common to all of the parameters $a, \ldots, e$, one can divide by it and obtain an equation which makes sense and leads to the states obtained above. For most of this work we shall take $m = 1/2$ for simplicity.

Let us now consider how our proposal applies to higher dimensions in the unmodified case. Taking the action to be

$$S = \sum_{j=1}^{N} \left| (\Delta x_j)^2(\Delta p_j)^2 - \frac{1}{4} \right| , \tag{4.26}$$

one finds the equation

$$\sum_{j=1}^{N} \mathcal{O}_{x_j,p_j}|\phi\rangle = 0 , \tag{4.27}$$

where the operators $\mathcal{O}_{x_j,p_j}$ are given by eq. (4.7). As these N operators commute, particular solutions are found in the intersection of their kernels. These are the usual coherent states in N dimensions. One can see from here that allowing in the action terms associated to couples like $\hat{x}_1, \hat{p}_2$ which commute would spoil this result.

5. High dimensional extensions

5.1 Generalized coherent states

The generalization of the notion of coherent states developed by [21] for non commutative theories has been considered by many authors [22]–[25]. Having a deformation of the usual position-momentum commutation relations, one constructs operators obeying the relation

$$[\tilde{a}, \tilde{a}^+] = F(\tilde{a}\tilde{a}^+) . \tag{5.1}$$

The coherent states are then defined to be the eigenstates of the modified destruction operators:

$$\tilde{a}|\xi\rangle = \xi|\xi\rangle . \tag{5.2}$$

The function F is constant in the usual case. When the deformed creation-destruction operators can be constructed from the usual ones (a, a^+) in the following way:

$$\tilde{a} = f(\hat{n}+1)a\,, \qquad n = a^+a\,, \tag{5.3}$$

a link exists between the functions f and F. This approach is mathematically useful in the sense that a decomposition of the unity is obtained quite easily and many properties of the usual coherent states survive. However, the link with the Heisenberg uncertainties is far from trivial.

One can define a deformed creation-destruction algebra, find the eigenstates of the destruction operators, compute the associated uncertainties and then try to see to which extent they are minimal. We here address the problem in a different way: we impose the minimization of the uncertainties and try to find the relevant states. This approach is more analytical than algebraic. In the usual case, the two constructions lead to the same result as shown above.

To avoid any confusion, the states obtained by our procedure will be called *quasi classical* or *squeezed* while the ones obtained trough a deformation of the destruction operator will be referred to as *coherent*.

5.2 The Non commutative plane

Let us define the variables $\hat{X}_k, \hat{P}_k$ linked to the ones given in eq. (2.4) by the relations

$$\hat{x}_k = \sqrt{\theta}\hat{X}_k\,, \quad \hat{p}_k = \frac{\hbar}{\sqrt{\theta}}\hat{P}_k\,. \tag{5.4}$$

Illustrating now our method with the values $m = n = 1/2$, the three non trivial commutation relations lead to the dimensionless action

$$\tilde{S} = \left|\Delta X_1 \Delta X_2 - \frac{1}{2}\right| + \left|\Delta X_1 \Delta P_1 - \frac{1}{2}\right| + \left|\Delta X_2 \Delta P_2 - \frac{1}{2}\right|. \tag{5.5}$$

Multiplying by an overall factor, and going back to the former variables, we obtain the action which will be used in this subsection:

$$S = \hbar\left|\Delta x_1 \Delta x_2 - \frac{1}{2}\theta\right| + \theta\left|\Delta x_1 \Delta p_1 - \frac{1}{2}\hbar\right| + \theta\left|\Delta x_2 \Delta p_2 - \frac{1}{2}\hbar\right|. \tag{5.6}$$

From a dimensional analysis, one could guess the form of the terms appearing in eq. (5.6) but not their relative weights. The introduction of generic dimensionless variables and the adoption of the democratic rule displayed in eq. (5.5) fixes in an unambiguous way the action displayed in eq. (5.6). Introducing constant factors to balance the different terms leads essentially to the same differential equation.

The discussion of section 2 has shown that there is no state on which this action vanishes. We now look for those on which it is extremal. The absolute values can be removed as in the previous section. Varying the action we obtain the field equation

$$\langle\psi|\psi\rangle\frac{\delta S}{\delta\psi^*(p)} = b_1\frac{\delta}{\delta\psi^*(p)}\Delta x_1 + b_2\frac{\delta}{\delta\psi^*(p)}\Delta x_2 + b_3\frac{\delta}{\delta\psi^*(p)}\Delta p_1 + b_4\frac{\delta}{\delta\psi^*(p)}\Delta p_2 = 0\,, \tag{5.7}$$

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where the coefficients b_i are given by the following expressions

$$b_1 = \hbar\Delta x_2 + \theta\Delta p_1\,, \quad b_2 = \hbar\Delta x_1 + \theta\Delta p_2\,, \quad b_3 = \theta\Delta x_1\,, \quad b_4 = \theta\Delta x_2\,. \tag{5.8}$$

Intermediate results such as

$$\begin{aligned}\frac{\delta}{\delta\psi^*(p)}\Delta x_1^2 &= \Big[-\hbar^2\partial_{p_1}^2 - i(\theta p_2 + 2\hbar\langle\hat{x}_1\rangle)\partial_{p_1} + \\ &\qquad + \left(\frac{1}{4}\frac{\theta^2}{\hbar^2}p_2^2 + \frac{\theta}{\hbar}\langle\hat{x}_1\rangle p_2 + \langle\hat{x}_1\rangle^2 - (\Delta x_1)^2\right)\Big]\psi(p)\,, \qquad (5.9)\\ \frac{\delta}{\delta\psi^*(p)}\Delta p_1^2 &= \left(p_1^2 - 2\langle\hat{p}_1\rangle p_1 + \langle\hat{p}_1\rangle^2 - (\Delta p_1)^2\right)\psi(p)\,, \qquad (5.10)\end{aligned}$$

are necessary to recast the equation in the following form

$$\begin{aligned}&\Big[\bar{a}_1\partial_{p_1}^2 + \bar{a}_2\partial_{p_2}^2 + i\left(\bar{a}_1\frac{\theta}{\hbar^2}p_2 + \bar{a}_3\right)\partial_{p_1} + i\left(-\frac{\theta}{\hbar^2}\bar{a}_2 p_1 + \bar{a}_4\right)\partial_{p_2} + \\ &\qquad + (\bar{a}_5 p_1^2 + \bar{a}_6 p_2^2 + \bar{a}_7 p_1 + \bar{a}_8 p_2 + \bar{a}_9)\Big]\psi(p) = 0\,. \qquad (5.11)\end{aligned}$$

The real numbers $\bar{a}_i$ are linked to the previous coefficients b_i by the following set of relations

$$\begin{aligned}\bar{a}_1 &= -\hbar^2\frac{b_1}{2\Delta x_1}\,, \quad \bar{a}_2 = -\hbar^2\frac{b_2}{2\Delta x_2}\,, \quad \bar{a}_3 = -\hbar\langle\hat{x}_1\rangle\frac{b_1}{\Delta x_1}\,, \quad \bar{a}_4 = -\hbar\langle\hat{x}_2\rangle\frac{b_2}{\Delta x_2}\,,\\ \bar{a}_5 &= \frac{1}{8}\frac{\theta^2}{\hbar^2}\frac{b_2}{\Delta x_2} + \frac{b_3}{2\Delta x_3}\,, \quad \bar{a}_6 = \frac{1}{8}\frac{\theta^2}{\hbar^2}\frac{b_1}{\Delta x_1} + \frac{b_4}{2\Delta p_2}\,,\\ \bar{a}_7 &= -\frac{\theta}{\hbar}\langle\hat{x}_2\rangle\frac{b_2}{2\Delta x_2} - \langle\hat{p}_1\rangle\frac{b_3}{2\Delta p_1}\,, \quad \bar{a}_8 = -\frac{\theta}{\hbar}\langle\hat{x}_1\rangle\frac{b_1}{2\Delta x_1} - \langle\hat{p}_2\rangle\frac{b_4}{\Delta p_2}\,,\\ \bar{a}_9 &= \left(\langle\hat{x}_1\rangle^2 - (\Delta x_1)^2\right)\frac{b_1}{2\Delta x_1} + \left(\langle\hat{x}_2\rangle^2 - (\Delta x_2)^2\right)\frac{b_2}{2\Delta x_2} + \left(\langle\hat{p}_1\rangle^2 - (\Delta p_1)^2\right)\frac{b_3}{2\Delta p_1} + \\ &\quad + \left(\langle\hat{p}_2\rangle^2 - (\Delta p_2)^2\right)\frac{b_4}{2\Delta p_2}\,. \qquad (5.12)\end{aligned}$$

To summarize, states which are extrema of the action given in eq. (5.6) satisfy the equation displayed in eq. (5.11). The real quantities appearing in this equation are linked to the observables of these state-solutions by the two sets of relations given in eqs. (5.8), (5.12). As done in the previous section, we introduce an operator depending on the state $|\psi\rangle$ and apply it to a state $|\phi\rangle$. This amounts to replace $\psi(p)$ by $\phi(p)$ in eq. (5.12) while the coefficients a_k depend on $|\psi\rangle$. This is the equivalent of eq. (4.7).

Before proceeding, let us point out one important difference between our approach and the usual generalization of coherent states. The operators $\hat{x}_i, \hat{p}_j$ of the non commutative plane can be written as linear combinations of some $\hat{Q}_j, \hat{P}_j$ which obey the usual commutation relations. As discussed in [21], the concept of coherent states is defined for the algebra and its representation. This means it can not be affected by a change of generators. On the contrary, our approach to what we call *squeezed* states depends on the generators, as shown by the mixing of p_1 and p_2 in eq. (5.11). This is hardly surprising since the action we began with, although it physically makes sense, is not invariant for linear transformations on the phase space. One can use the operators $\hat{Q}_j, \hat{P}_j$ to show the impossibility to saturate simultaneously the three uncertainties analyzed in section 2; we simply restricted ourselves to a reasoning which could also be used for the other models where the link to an undeformed algebra is less obvious.

The change of variables performed in eq. (5.4) is not well defined in the limit $\theta \longrightarrow 0$ and the action S does not coincide with two copies of the one dimensional case displayed in eq. (4.2)(generalized coherent states also become singular when the non parameter controlling non commutativity is sent to zero). However, the differential equation it generates has that nice property. Looking at eq. (5.11), one sees that the factors multiplying the terms $p_1\partial_{p_2}$ and $p_2\partial_{p_1}$ go to zero in that limit so that one obtains the sum of two separate operators $\mathcal{O}_{x_1,p_1}$ and $\mathcal{O}_{x_2,p_2}$ just like in eq. (4.27). Using eqs. (5.8), (5.12), one sees that the coefficients $\bar{a}_1, \bar{a}_2$ of the derivatives $\partial^2_{p_1}, \partial^2_{p_2}$ admit expansions in the deformation parameter θ. This means a perturbation theory in this context will not be straightforward.

As in the previous section, we will take quantities a_k depending on a state $|\psi\rangle$ and write the associated differential equation for a state $|\phi\rangle$. We don't have the most general solution to this second order partial differential equation. We shall look for special cases in which explicit solutions can be found. It is rather interesting that the states found in eq.(3.9)–(3.10) which display periodic behaviors for the harmonic oscillator are particular solutions of the aforementioned equation.

It is obvious that the coefficients b_i are positive. As a consequence, one has

$$\bar{a}_1, \bar{a}_2 < 0\,, \qquad \bar{a}_5, \bar{a}_6 > 0\,. \tag{5.13}$$

We shall take into account the signs of these coefficients by the following parametrization: $\bar{a}_1 = -a_1^2, \bar{a}_2 = -a_2^2, \bar{a}_5 = a_5^2, \bar{a}_6 = a_6^2$. To simplify future formulas, we assume from now on $\hbar = \theta = 1$. Introducing the variables y_1, y_2 by the relations

$$p_1 = a_1 y_1 - \frac{a_4}{a_2^2}\,, \qquad p_2 = a_2 y_2 + \frac{a_3}{a_1^2}\,, \tag{5.14}$$

our differential equation takes the simpler form

$$\left[-\partial^2_{y_1} - \partial^2_{y_2} + i t_1^2(-y_2\partial_{y_1} + y_1\partial_{y_2}) + (t_2^2 y_1^2 + t_3^2 y_2^2 + t_4 y_1 + t_5 y_2 + t_6)\right]\phi(y) = 0\,, \tag{5.15}$$

where

$$t_1^2 = a_1 a_2\,, \quad t_2^2 = a_1^2 a_5^2\,, \quad t_3^2 = a_2^2 a_6^2\,, \quad t_4 = \left(-2\frac{a_5^2}{a_2^2}a_4 + a_7\right)a_1\,, \quad t_5 = \left(2\frac{a_6^2}{a_1^2}a_3 + a_8\right)a_2\,,$$
$$t_6 = a_9 + \left(\frac{a_4 a_5}{a_2^2}\right)^2 - \frac{a_4 a_7}{a_2^2} + \left(\frac{a_3 a_6}{a_1^2}\right)^2 + \frac{a_3 a_8}{a_1^2}\,. \tag{5.16}$$

5.2.1 The gaussian solution

Similarly to the one dimensional case, one can, *in this case*, postulate a solution which is the exponential of a quadratic function:

$$\psi(p) = N \exp\left(z_1 y_1^2 + z_2 y_2^2 + z_3 y_1 y_2 + z_4 y_1 + z_5 y_2\right). \tag{5.17}$$

Plugging this wave function in the differential equation, one ends up with the relations

$$
\begin{aligned}
z_3 &= \frac{t_1^2}{4(t_2^2 - t_3^2)} \left(\sqrt{-(t_1^2 - 4t_2^2)(t_1^2 - 4t_3^2)} + i(t_1^4 - 2(t_2^2 + t_3^2)) \right) , \\
z_1 &= \frac{1}{2}\sqrt{-z_3^2 - iz_3 t_1^2 + t_2^2} , \\
z_2 &= \frac{1}{2}\sqrt{-z_3^2 - iz_3 t_1^2 + t_3^2} , \quad z_5 = \frac{2z_3 t_4 + it_1^2 t_4 - 4z_1 t_5}{16 z_1 z_2 - 4z_3^2 - t_1^4} , \quad z_4 = -\frac{2z_3 z_5 - iz_5 t_1^2 - t_4}{4z_1} , \\
&\quad 2z_1 + 2z_2 + z_4^2 + z_5^2 - t_6 = 0 .
\end{aligned} \tag{5.18}
$$

Each coefficient has been written solely in terms of those appearing before it in the list and the first one, z_3, depends only on the parameters of the equation. The last formula of eq. (5.18) is a constraint which has to be satisfied for the differential equation to admit a gaussian solution; this is reminiscent of eq. (4.12) in the usual one dimensional case. The normalizability of the state imposes conditions on z_1, z_2, z_3.

5.2.2 The cylindrically symmetric solution

In the preceding subsection we saw that a carefully chosen relation between the coefficients of the differential equation led to an explicit solution. Although in the usual one dimensional case such a choice was ultimately justified because it led to the absolute minimum of the action, nothing like that occurs here. One has to resort to a second order analysis to find the true nature of the critical points represented by the states obtained so far. That will not be done here; we simply find some explicit solutions.

There is a set of simple relations between the parameters of the equation which allows a cylindrical separation of variables. In fact, if

$$
t_3 = t_2 , \qquad t_4 = t_5 = 0 , \tag{5.19}
$$

the parameterization $\phi = R(r)e^{im\theta}$ in the polar coordinates linked to the cartesian coordinates (y_1, y_2) leads to the ordinary differential equation

$$
R''(r) + \frac{1}{r}R'(r) + \left(\frac{m^2}{r^2} + (t_6 - mt_1^2) + t_2^2 r^2 \right) R(r) = 0 . \tag{5.20}
$$

In the special case where the relation $t_6 = mt_1^2$ holds, the solution can be re casted as a combination of two Bessel functions:

$$
R(r) = C_1 I_{-\frac{m}{2}} \left(\frac{t_2 r^2}{2} \right) + C_2 I_{\frac{m}{2}} \left(\frac{t_2 r^2}{2} \right) . \tag{5.21}
$$

5.2.3 A third explicit solution

Let us consider the case

$$
(t_1, t_3, t_4, t_5, t_6) = (2, 2, 0, 0, -4) , \quad \text{with} \quad t_2 \quad \text{arbitrary} . \tag{5.22}
$$

It is straightforward to verify that for any integer m and any complex constants c_{01}, c_{11}, the function

$$
\phi(r, \theta) = r^m \exp\left[-c_{01} e^{i\theta} r - r^2 \left(1 + \frac{1}{4} c_{11} e^{2i\theta} \right) + im\theta \right] \tag{5.23}
$$

is a solution of eq. (5.15).

5.2.4 Factorizability

As in the undeformed theory, we can look for the particular conditions under which the second order differential operator can be written as a product. This is found to occur when

$$t_2 = t_3 = \frac{t_1^2}{2}\,, \qquad t_6 = t_1^2\,, \qquad t_4 = t_5 = 0\,. \tag{5.24}$$

The operators k_1, k_2 assume the forms

$$k_1 = \left(i\partial_{y_1} + \partial_{y_2} - \frac{i}{2}t_1^2(y_1 - iy_2)\right)\,, \quad k_2 = \left(i\partial_{y_1} - \partial_{y_2} + \frac{i}{2}t_1^2(y_1 + iy_2)\right)\,. \tag{5.25}$$

This case of reducibility forms a particular subset which is at the intersection of the ones corresponding to gaussian solutions and the ones displaying polar symmetry; it does not bring in new solutions.

5.3 The model with a minimal uncertainty in length

We now restrict to the second extension of the KMM model corresponding to the following choice of the functions appearing in eq. (2.11):

$$g(\vec{p}^2) = \beta\,, \qquad f(\vec{p}^2) = \frac{\beta\vec{p}^2}{-1 + \sqrt{1 + 2\beta\vec{p}^2}}\,. \tag{5.26}$$

The main interest of this model lies in the fact that its Q.F.T. is finite [4]. It admits a representation in which the operators look much simpler than in eq. (2.14):

$$\hat{x}_i = i\hbar\partial_{\rho_i}\,, \quad p_i = \frac{\rho_i}{1 - \frac{1}{2}\beta\vec{\rho}^2}\,; \tag{5.27}$$

the scalar product is the usual one, but now it is defined on the disk of radius $\sqrt{\frac{2}{\beta}}$. The couples which enter the action S are $(x_1, p_1), (x_1, p_2), (x_2, p_1)$ and (x_2, p_2). The reasoning we have used in the preceding sections lead to the final equation

$$\begin{aligned} 0 = \Bigg[&\left(1 - \frac{1}{2}\beta\vec{\rho}^2\right)^2 (-a_1^2\partial_{\rho_1}^2 - a_2^2\partial_{\rho_2}^2) + \frac{2i}{\hbar}\left(1 - \frac{1}{2}\beta\vec{\rho}^2\right)(-a_1^2\langle\hat{x}_1\rangle\partial_{\rho_1} - a_2^2\langle\hat{x}_2\rangle\partial_{\rho_2}) + \\ &+ a_3\rho_1^2 + a_4\rho_2^2 + a_5\rho_1\rho_2 + (a_6\rho_1 + a_7\rho_2)\left(1 - \frac{1}{2}\beta\vec{\rho}^2\right) + a_8\left(1 - \frac{1}{2}\beta\vec{\rho}^2\right)^2\Bigg]\psi(\rho_1, \rho_2)\,. \end{aligned} \tag{5.28}$$

The symmetric case $a_2 = a_1$, $\langle x_1\rangle = \langle x_2\rangle = 0$, $a_3 = a_4 = a_5 = a_6 = a_7 = 0$ admits a solution which is a combination of Bessel functions multiplied by phases:

$$\psi(\rho) = \left(C_1 I_{-m}\left(\frac{\sqrt{-a_8}}{a_1}\rho\right) + C_2 I_m\left(\frac{\sqrt{-a_8}}{a_1}\rho\right)\right)e^{im\theta}\,. \tag{5.29}$$

The coefficients a_6, a_7 and a_8 vanish with β.

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6. A look at the fuzzy sphere

The fuzzy sphere is a matrix model defined by the following relations [25]:

$$[\hat{x}_k, \hat{x}_l] = \frac{i\alpha}{\sqrt{j(j+1)}} \epsilon_{klm} \hat{x}_m \, ; \qquad \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1 \, , \quad \text{with} \quad j \quad \text{half} - \text{integer} \, . \tag{6.1}$$

The saturation of the uncertainties related to the pairs of non commuting variables translate into the formulas $m_{jk}|\psi\rangle = 0$, with

$$m_{12} = \hat{x}_1 + ia\hat{x}_2 + (d + ie) \, , \quad m_{23} = \hat{x}_2 + ib\hat{x}_3 + (f + ge) \, , \text{ and } \quad m_{31} = \hat{x}_3 + ic\hat{x}_1 + (h + ik) \, , \tag{6.2}$$

where $a, b, c, \cdots$ are real. Considering the following combinations of these equations

$$\begin{aligned} (c[m_{12}, m_{23}] - i[m_{23}, m_{31}])|\psi\rangle &= 0 \, , \\ (-i[m_{12}, m_{23}] + b[m_{31}, m_{12}])|\psi\rangle &= 0 \, , \\ (-a[m_{23}, m_{31}] + i[m_{31}, m_{12}])|\psi\rangle &= 0 \, , \end{aligned} \tag{6.3}$$

one obtains(in units where $\alpha = 1$)

$$(1 - iabc)\hat{x}_1|\psi\rangle = 0 \, , \qquad (1 - iabc)\hat{x}_3|\psi\rangle = 0 \, , \qquad (1 - iabc)\hat{x}_2|\psi\rangle = 0 \, . \tag{6.4}$$

Can the three Heisenberg inequalities be saturated simultaneously? Only two cases may lead to that situation:

- The first possibility is
$$\hat{x}_1|\psi\rangle = \hat{x}_2|\psi\rangle = \hat{x}_3|\psi\rangle = 0 \, , \tag{6.5}$$
but then the second part of eq. (6.1) is violated .
- The remaining possibility
$$1 - iabc = 0 \tag{6.6}$$
can be rewritten as
$$\frac{\langle \hat{x}_1 \rangle \langle \hat{x}_2 \rangle \langle \hat{x}_3 \rangle}{(\Delta x_1)^2 (\Delta x_2)^2 (\Delta x_3)^2} = -8i \, ; \tag{6.7}$$
this is not possible since all the quantities on the left side are real.

The method proposed here can in principle be applied to the fuzzy sphere. In this section, we work with an action slightly different from the ones used so far, choosing $m = 1$:

$$S = \left[\left((\Delta x_1)^2 (\Delta x_2)^2 - \frac{1}{2} \langle x_3 \rangle^2 \right)^2 + \text{permutations} \right] . \tag{6.8}$$

One important feature which distinguishes the fuzzy sphere from the models we have studied before is the fact that its Fock space is finite dimensional. This results in the fact that the action S is now a function rather than a functional.

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To illustrate how our approach applies to the fuzzy sphere, let us take the very simple case $j = 1$. A unitary transformation is performed to go from the usual representation to a more symmetric one in which the non vanishing elements of the operators are

$$(X_1)_{23} = (X_1)_{32} = 1\,, \quad (X_2)_{13} = -(X_2)_{31} = -i\,, \quad (X_3)_{12} = (X_3)_{21} = 1\,. \tag{6.9}$$

The Fock space in this trivial case is six dimensional on the real scalars. Any state can be written as

$$|\psi\rangle = (z_1, z_2, z_3, z_4, z_5, z_6)\,. \tag{6.10}$$

The mean values we need in order to compute the action can be written as

$$\langle x_k \rangle = \frac{N_k}{G}\,, \qquad \langle x_k^2 \rangle = \frac{K_k}{G}\,, \tag{6.11}$$

where

$$\begin{aligned} N_1 &= 2(z_3 z_5 + z_4 z_6)\,, \quad N_2 = 2(-z_2 z_5 + z_1 z_6)\,, \quad N_3 = 2(z_1 z_3 + z_2 z_4)\,, \\ K_1 &= z_3^2 + z_4^2 + z_5^2 + z_6^2\,, \quad K_2 = z_1^2 + z_2^2 + z_5^2 + z_6^2\,, \quad K_3 = z_1^2 + z_2^2 + z_3^2 + z_4^2\,, \\ G &= z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2\,. \end{aligned} \tag{6.12}$$

The interest of the representation displayed by eq. (6.9) is that it makes the symmetries of the problem more transparent. One easily verifies that the transformation

$$\tau(z_1, z_2, z_3, z_4, z_5, z_6) = (z_3, z_4, z_5, z_6, -z_2, z_1) \tag{6.13}$$

generates a circular permutation(discarding the signs) of the functions N_k. The same holds for the functions K_k while G remains unchanged so that τ is a symmetry of the action and, in addition, it is idempotent: $\tau^6 = -1$. Finding the extrema of the action given above is not an easy task. We give here two such extrema, obtained numerically using Mathematica:

$$\begin{aligned} &\vec{z} = (-1, 0, 0, 1, 1, 0) \quad \text{with} \quad S = \frac{16}{27} \quad \text{and} \\ &\vec{z} = (0.631646, 0.315353, 0.002528, 0.016494, -0.316359, 0.633415) \quad \text{with} \quad S = 1.19\,10^{-8}\,. \end{aligned} \tag{6.14}$$

Applying the transformation τ to these states, one generates states having the same values of S.

7. Conclusions

In this work, we have shown that the defining properties of the coherent states in the usual theory are lost in some non commutative models. This led us to suggest an approach toward squeezed states which relies on a functional. We have found special solutions to the second order differential equations obtained in two different non commutative theories. We have briefly outlined how our method can be applied to finite dimensional matrix models like the fuzzy sphere.

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The problem for the first two theories we have studied is that there are too many solutions to the relevant equations. On the contrary, for the fuzzy sphere, the situation is different; the action is a function of a finite number of variables but its form is non trivial and makes the search for solutions cumbersome.

One of the crucial points which remain to be addressed is the nature of the critical points found here. To know if these states are maxima, minima or saddle points of the action, one has to resort to a second order analysis. However, as the most general solutions of the second order partial differential equations involved are not known, such a computation cannot tell us by itself if we are in front of an absolute minimum. One can also develop the differential equations we obtained order by order in the extra parameter θ; this breaks the symmetry with $\hbar$ and makes any statement about the nature of the solution more difficult. One important difference with usual perturbation theory lies in the fact that the deformation parameter appears also multiplied by derivative operators. This is likely to be tackled by a treatment such as the one used in [15].

Some questions are of particular importance for the approach we suggest to be really valuable. For example, one would like to know if the states we obtained form an over complete system. If If this was the case, they might be legitimate candidates for the definition of a physically meaningful star product [26]. The fact that some solutions obtained here are eigenfunctions of Sturm-Liouville systems is promising.

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